

**IMPLICIT HAMILTONIAN SYSTEMS WITH SYMMETRY**

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Implicit Hamiltonian systems with symmetry are treated by exploiting the notion of symmetry of Dirac structures. It is shown how Dirac structures can be reduced to Dirac structures on the orbit space of the symmetry group, leading to a reduced implicit (generalized) Hamiltonian system. The approach is specialized to nonholonomic mechanical systems with symmetry.

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**1. Introduction**

The theory of mechanical systems with symmetry has a long and rich history. Most of the techniques appear in their classical form in Whittaker [25], while the modern “geometric” approach is due to several authors, including Marsden and Weinstein [14]; see especially [1, 19, 13] for excellent treatments. Within a Hamiltonian formulation the basic idea is that if there exists a, say abelian, symmetry group acting by canonical transformations on the phase space which leaves the Hamiltonian (total energy) of the mechanical system invariant, then the equations of motion may be reduced to the lower-dimensional space of orbits of the symmetry group. Furthermore, by Noether’s theorem, this reduced dynamics possesses conserved quantities (first integrals) directly related to the group action, whose existence admits a further reduction of the equations of motion. In this way the study of the dynamics of the mechanical system has been reduced to the study of a lower-order (still Hamiltonian) dynamics, since in some sense the full-order dynamics can be *reconstructed* from the lower-order dynamics. This is of obvious interest for analysis, but also for control and simulation purposes.

Recently, there has been a revival of interest for mechanical systems subject to nonholonomic kinematic constraints, as arising e.g. from non-slipping conditions. Such constraints are frequently encountered in mechanisms and robotic systems (see e.g.

[17] for a beautiful classical reference). One of the aims of the recent work in this area, see e.g. [9, 2, 4, 6], is to clarify the relation between the existence of symmetry for such systems (e.g. rotational invariance) and the possibilities for reduction. The main obstacle is the fact that systems with nonholonomic kinematic constraints cannot be cast into the standard Lagrangian or Hamiltonian setting, and thus appropriate generalizations of these frameworks have to be sought for. For the Lagrangian side this has been pursued e.g. in [9, 4], while the description of nonholonomic systems as generalized Hamiltonian systems has been undertaken e.g. in [2, 6, 12, 22] (see also [10] for the relation between the Lagrangian and Hamiltonian approach).

In our previous work [22, 15, 20, 21] we have shown that not only nonholonomic mechanical systems give rise to a generalized Hamiltonian formulation, but other energy-conserving physical systems (such as electrical *LC*-circuits) as well. Furthermore, it has been argued in [3, 7, 20, 21, 23] that a proper Hamiltonian formulation of all such systems can be based on the geometric notion of a (generalized) Dirac structure, as introduced as a generalization of Poisson and symplectic structures by Courant [5] and Dorfman [8]. In fact, the concept of a Dirac structure allows to give a simple intrinsic definition of an *implicit* (generalized) Hamiltonian system, that is, a mixed set of differential and algebraic equations of “Hamiltonian form” as frequently encountered in modelling. From a physical point of view the Dirac structure seems to capture naturally the geometric structure of the system as arising from the interconnection of simple subsystems [7, 23, 16, 3].

The purpose of the present paper is to treat a notion of *symmetry* for (generalized) Dirac structures and general implicit Hamiltonian systems, which properly generalizes the notion of symmetry for symplectic and Poisson structures and (standard) Hamiltonian systems. A basic starting point herein is the definition of a symmetry of a Dirac structure given by Dorfman [8], see also [5]. Further, we deduce some basic results on the characterization of such symmetries and the *reduction* of generalized Dirac structures and implicit Hamiltonian systems, as well as a few results on the relation with conserved quantities. These general results will then be applied to the particular case of nonholonomic mechanical systems, leading to the study of the same type of symmetries as considered in the previous papers [9, 2, 4, 6]. This will be done in Section 3 after a concise treatment of generalized Dirac structures and implicit Hamiltonian systems in Section 2. Finally, in Section 4 we illustrate our approach on three simple examples; two of which are in the realm of nonholonomic systems and have been treated before in [2, 4]. Conclusions follow in Section 5.

## 2. Dirac structures and implicit Hamiltonian systems

The notion of *Dirac structures* has been introduced by Courant [5] and Dorfman [8] as a generalization of symplectic and Poisson structures. Let  $\mathcal{X}$  be a manifold with the tangent bundle  $T\mathcal{X}$  and the co-tangent bundle  $T^*\mathcal{X}$ . We define  $T\mathcal{X} \oplus T^*\mathcal{X}$  as the smooth vector bundle over  $\mathcal{X}$  with the fibre at each  $x \in \mathcal{X}$  given by  $T_x\mathcal{X} \times T_x^*\mathcal{X}$ . Let  $X$  be a smooth vector field and  $\alpha$  a smooth one-form on  $\mathcal{X}$ , respectively. Given a smooth vector subbundle  $\mathcal{D} \subset T\mathcal{X} \oplus T^*\mathcal{X}$ , we say that the pair  $(X, \alpha)$  belongs to  $\mathcal{D}$

(denoted  $(X, \alpha) \in \mathcal{D}$ ) if  $(X(x), \alpha(x)) \in \mathcal{D}(x)$  for every  $x \in \mathcal{X}$ . Furthermore we define the smooth vector subbundle  $\mathcal{D}^\perp \subset T\mathcal{X} \oplus T^*\mathcal{X}$  as

$$\mathcal{D}^\perp = \{(X, \alpha) \in T\mathcal{X} \oplus T^*\mathcal{X} \mid \langle \alpha \mid \hat{X} \rangle + \langle \hat{\alpha} \mid X \rangle = 0, \quad \forall (\hat{X}, \hat{\alpha}) \in \mathcal{D}\} \tag{1}$$

with  $\langle \cdot \mid \cdot \rangle$  denoting the duality inner product between a one-form and a vector field. In (1) and throughout in the sequel the pairs  $(X, \alpha), (\hat{X}, \hat{\alpha})$  are assumed to be pairs of *smooth* vector fields and *smooth* one-forms.

DEFINITION 1 [5, 8, 7]. A generalized Dirac structure on a manifold  $\mathcal{X}$  is a smooth vector subbundle  $\mathcal{D} \subset T\mathcal{X} \oplus T^*\mathcal{X}$  such that  $\mathcal{D}^\perp = \mathcal{D}$ . A Dirac structure is a generalized Dirac structure  $\mathcal{D}$  satisfying the *closedness* (or *integrability*) condition

$$\langle L_{X_1}\alpha_2 \mid X_3 \rangle + \langle L_{X_2}\alpha_3 \mid X_1 \rangle + \langle L_{X_3}\alpha_1 \mid X_2 \rangle = 0 \tag{2}$$

for all  $(X_1, \alpha_1), (X_2, \alpha_2), (X_3, \alpha_3) \in \mathcal{D}$ .

EXAMPLE 1 [5, 8, 7]. Let  $\{, \}$  be a Poisson bracket on  $\mathcal{X}$  with the structure matrix  $J(x)$ . Then the graph of  $J(x)$ , that is,  $\mathcal{D} = \{(X, \alpha) \in T\mathcal{X} \oplus T^*\mathcal{X} \mid X(x) = J(x)\alpha(x), x \in \mathcal{X}\}$ , is a Dirac structure on  $\mathcal{X}$ . The Jacobi identity for  $\{, \}$  is equivalent to (2).

EXAMPLE 2 [5, 8, 7]. Let  $\omega$  be a two-form on  $\mathcal{X}$ . Then  $\mathcal{D} = \{(X, \alpha) \in T\mathcal{X} \oplus T^*\mathcal{X} \mid i_X\omega = \alpha\}$  is a generalized Dirac structure on  $\mathcal{X}$ , which satisfies (2) if and only if  $d\omega = 0$ .

EXAMPLE 3 [7]. Let  $G$  be a smooth constant-dimensional distribution on  $\mathcal{X}$ , and let  $\text{ann } G$  be its annihilating smooth co-distribution. Then  $\mathcal{D} = \{(X, \alpha) \in T\mathcal{X} \oplus T^*\mathcal{X} \mid X \in G, \alpha \in \text{ann } G\}$  defines a generalized Dirac structure on  $\mathcal{X}$ , which satisfies (2) if and only if  $G$  is involutive.

DEFINITION 2 [20, 21, 7]. Let  $\mathcal{X}$  be a manifold with (generalized) Dirac structure  $\mathcal{D}$ , and let  $H : \mathcal{X} \rightarrow \mathbb{R}$  be a smooth function (the Hamiltonian). The *implicit (generalized) Hamiltonian system* corresponding to  $(\mathcal{X}, \mathcal{D}, H)$  is given by the specification

$$(\dot{x}, dH(x)) \in \mathcal{D}(x), \quad x \in \mathcal{X}. \tag{3}$$

REMARK 3. By substituting  $\alpha = \hat{\alpha} = dH(x)$ , and  $X = \hat{X} = \dot{x}$  in (1) one immediately obtains for every implicit generalized Hamiltonian system the energy-conservation property  $\frac{dH}{dt} = \langle dH(x) \mid \dot{x} \rangle = 0$ .

Note that (3) describes in general a *mixed* set of differential and algebraic equations (DAE's) of the form  $F(\dot{x}, x) = 0$ . If the Dirac structure is defined by a Poisson bracket with the structure matrix  $J$  as in Example 1, then (3) reduces to the (explicit) Hamiltonian system

$$\dot{x} = J(x) \frac{\partial H}{\partial x}(x) \tag{4}$$

with  $\frac{\partial H}{\partial x}(x)$  denoting the column vector of partial derivatives of  $H$ . In [7, 23, 16] it has been shown that power-conserving interconnections of conservative mechanical

systems naturally lead to implicit generalized Hamiltonian systems as in (3), which in general are *not* of the explicit form as in (4).

In [7], expanding on [5], different ways of representing (generalized) Dirac structures and implicit (generalized) Hamiltonian systems have been introduced. We recall the following two representations. First, we associate with a generalized Dirac structure  $\mathcal{D}$  on  $\mathcal{X}$  the smooth distributions

$$\begin{aligned} G_0 &:= \{X \in T\mathcal{X} \mid (X, 0) \in \mathcal{D}\}, \\ G_1 &:= \{X \in T\mathcal{X} \mid \exists \alpha \in T^*\mathcal{X} \text{ s.t. } (X, \alpha) \in \mathcal{D}\}, \end{aligned} \quad (5)$$

and the smooth co-distributions

$$\begin{aligned} P_0 &:= \{\alpha \in T^*\mathcal{X} \mid (0, \alpha) \in \mathcal{D}\}, \\ P_1 &:= \{\alpha \in T^*\mathcal{X} \mid \exists X \in T\mathcal{X} \text{ s.t. } (X, \alpha) \in \mathcal{D}\}. \end{aligned} \quad (6)$$

It immediately follows that  $G_0 \subset G_1$ ,  $P_0 \subset P_1$ , while by  $\mathcal{D} = \mathcal{D}^\perp$  one obtains [7]

$$\begin{aligned} G_0 &= \ker P_1 \\ P_0 &= \text{ann } G_1. \end{aligned} \quad (7)$$

If  $\mathcal{D}$  satisfies the closedness condition (2), then (cf. [8]) the (co-)distributions  $G_0, G_1, P_0, P_1$  are all *involutive*.

REMARK 4. The distribution  $G_1$  describes the set of admissible flows of any implicit generalized Hamiltonian system corresponding to  $\mathcal{D}$ . In particular, if  $G_1$  is constant-dimensional and involutive then we may find by Frobenius' theorem local coordinates  $(x_1, \dots, x_n)$  for  $\mathcal{X}$  such that  $P_0 = \text{span}\{dx_1, \dots, dx_k\}$ , implying that  $x_1, \dots, x_k$  are independent conserved quantities for (3). Dually, the co-distribution  $P_1$  describes, together with the Hamiltonian  $H : \mathcal{X} \rightarrow \mathbb{R}$ , the algebraic constraints of the implicit generalized Hamiltonian system (3), that is

$$dH(x) \in P_1(x), \quad x \in \mathcal{X}. \quad (8)$$

REMARK 5. A (generalized) Dirac structure is of the type as described in Example 3 if and only if  $G_0 = G_1 =: G$ , with  $G$  constant-dimensional.

THEOREM 6 [7]

(a) Let  $\mathcal{D}$  be a generalized Dirac structure on  $\mathcal{X}$ , with  $P_1$  constant-dimensional. Then there exists a skew-symmetric linear map

$$J(x) : P_1(x) \subset T_x^*\mathcal{X} \longrightarrow (P_1(x))^* \simeq T_x\mathcal{X}/G_0(x) \quad (9)$$

with kernel  $P_0(x)$  such that

$$\mathcal{D} = \{(X, \alpha) \mid X(x) - J(x)\alpha(x) \in \ker P_1(x), x \in \mathcal{X}, \alpha \in P_1\}. \quad (10)$$

Conversely, define  $\mathcal{D}$  for any skew-symmetric linear map  $J(x) : T_x^*\mathcal{X} \rightarrow T_x\mathcal{X}$  and

constant-dimensional co-distribution  $P_1$  as in (10), then  $\mathcal{D}$  is a generalized Dirac structure on  $\mathcal{X}$ .

- (b) Let  $\mathcal{D}$  be a generalized Dirac structure on  $\mathcal{X}$ , with  $G_1$  constant-dimensional. Then there exists a skew-symmetric linear map

$$\omega(x) : G_1(x) \subset T_x\mathcal{X} \rightarrow (G_1(x))^* \simeq T_x^*\mathcal{X}/P_0(x) \quad (11)$$

with kernel  $G_0(x)$  such that

$$\mathcal{D} = \{(X, \alpha) \mid \alpha(x) - \omega(x)X(x) \in \text{ann } G_1(x), x \in \mathcal{X}, X \in G_1\}. \quad (12)$$

Conversely, define  $\mathcal{D}$  for any skew-symmetric linear map  $\omega(x) : T_x\mathcal{X} \rightarrow T_x^*\mathcal{X}$  and constant-dimensional distribution  $G_1$  as in (12), then  $\mathcal{D}$  is a generalized Dirac structure on  $\mathcal{X}$ .

Representation (a) of the generalized Dirac structure yields the following local representation of the implicit generalized Hamiltonian system (3):

$$\begin{aligned} \dot{x} &= J(x) \frac{\partial H}{\partial x}(x) + g(x)\lambda, \\ 0 &= g^T(x) \frac{\partial H}{\partial x}(x), \end{aligned} \quad (13)$$

where the full-rank matrix  $g(x)$  has been chosen such that  $\text{Im } g(x) = G_0(x) = \ker P_1(x)$ , and  $J(x)$  in (9) has been arbitrarily extended to a skew-symmetric map  $T_x^*\mathcal{X} \rightarrow T_x\mathcal{X}$ . Here the vector  $\lambda$  are Lagrange multipliers corresponding to the algebraic constraints  $0 = g^T(x) \frac{\partial H}{\partial x}(x)$ ; under nondegeneracy conditions on  $H$  they will be uniquely determined (see the discussion later on). Analogously, representation (b) of the generalized Dirac structure yields the following local representation of the implicit generalized Hamiltonian system (3)

$$\begin{aligned} \frac{\partial H}{\partial x}(x) &= \omega(x)\dot{x} + p(x)\lambda, \\ 0 &= p^T(x)\dot{x}, \end{aligned} \quad (14)$$

where the full-rank matrix  $p(x)$  is such that  $\text{Im } p(x) = P_0(x) = \text{ann } G_1(x)$ .

EXAMPLE 4 [7]. A classical mechanical system with Hamiltonian  $H(q, p)$  subject to  $k$  independent kinematic constraints  $A^T(q)\dot{q} = 0$  (with  $A^T(q)$  of full row-rank) can be either written as in representation (a)

$$\begin{aligned} \begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} &= \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial q}(q, p) \\ \frac{\partial H}{\partial p}(q, p) \end{bmatrix} + \begin{bmatrix} 0 \\ A(q) \end{bmatrix} \lambda, \\ 0 &= [0 \ A^T(q)] \begin{bmatrix} \frac{\partial H}{\partial q}(q, p) \\ \frac{\partial H}{\partial p}(q, p) \end{bmatrix}, \end{aligned} \quad (15)$$

or as in representation (b)

$$\begin{aligned} \begin{bmatrix} \frac{\partial H}{\partial q}(q, p) \\ \frac{\partial H}{\partial p}(q, p) \end{bmatrix} &= \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix} \begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix} + \begin{bmatrix} A(q) \\ 0 \end{bmatrix} \lambda, \\ 0 &= [A^T(q) \quad 0] \begin{bmatrix} \dot{q} \\ \dot{p} \end{bmatrix}. \end{aligned} \quad (16)$$

(Note that in this case the Lagrange multipliers  $\lambda$  have the physical interpretation of being constraint forces!). The underlying generalized Dirac structure satisfies the closedness condition (2) if and only if the kinematic constraints  $A^T(q)\dot{q} = 0$  are *holonomic* [7, 22].

*Remark.* The generalized Hamiltonian representation of systems with kinematic constraints as proposed in [2, 6] *combines* in some sense (15) and (16), by noting that the symplectic form  $\omega = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$  in (16) is *nondegenerate* when restricted to the tangent space of the constraint manifold defined by the last equations of (15) *intersected* with the distribution defined by the last equations of (16), provided the Hamiltonian  $H$  is as in Remark 11 below.

A coordinate-free description of the underlying (generalized) Dirac structure in Example 4 can be given as follows. Let  $q$  be local coordinates for the configuration manifold  $Q$ . The rows of the matrix  $A^T(q)$  are local coordinate expressions of independent one-forms  $\alpha_1, \dots, \alpha_k$  on  $Q$ . The cotangent bundle  $T^*Q$  is endowed with the natural symplectic form  $\omega$  (and  $(q, p)$  are canonical coordinates with respect to  $\omega$ ), yielding a bundle isomorphism  $\omega : TT^*Q \rightarrow T^*T^*Q$ , also denoted by  $\omega$ . Define the co-distribution  $P_0 := \text{span}\{\pi^*\alpha_1, \dots, \pi^*\alpha_k\}$  (with  $\pi : T^*Q \rightarrow Q$  the natural projection), let  $G_1 := \ker P_0$ , and define the generalized Dirac structure as in (11) by restricting  $\omega(x)$  to  $G_1(x)$ . Note that

$$G_0 = \omega^{-1}(P_0). \quad (17)$$

Summarizing, we have the following intrinsic characterization of the (generalized) Dirac structure in Example 4.

**PROPOSITION 7.** *Let  $\alpha_1, \dots, \alpha_k$  be independent one-forms on  $Q$ . Let  $\omega$  be the canonical 2-form on  $T^*Q$ . Define the co-distribution  $P_0 := \text{span}\{\pi^*\alpha_1, \dots, \pi^*\alpha_k\}$  on  $T^*Q$ , with  $\pi : T^*Q \rightarrow Q$  the natural projection. Then  $\mathcal{D}$  defined as in (11) is a generalized Dirac structure on  $\mathcal{X} = T^*Q$ , which satisfies the closedness condition (2) iff  $P_0$  is involutive (cf. [7]).*

For a Dirac structure, that is, a generalized Dirac structure satisfying the closedness condition (2), we can in some sense combine representations (a) and (b). In fact, see

[5, 8], around every point  $x_0$  where  $G_1$  and  $P_1$  have constant dimension, condition (2) will be satisfied if and only if there exist local (canonical) coordinates

$$(q, p, r, s) = (q_1, \dots, q_k, p_1, \dots, p_k, r_1, \dots, r_\ell, s_1, \dots, s_m) \quad (18)$$

about  $x_0$  such that

$$\begin{aligned} \mathcal{D}(q, p, r, s) = \{ (X, \alpha) \mid X = (X^q, X^p, X^r, X^s), \alpha = (\alpha^q, \alpha^p, \alpha^r, \alpha^s), \\ X^q = \alpha^p, X^p = -\alpha^q, X^r = 0, \alpha^s = 0 \}. \end{aligned} \quad (19)$$

In these coordinates the implicit Hamiltonian system takes the simple form

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p}(q, p, r, s), \\ \dot{p} &= -\frac{\partial H}{\partial q}(q, p, r, s), \\ \dot{r} &= 0, \\ 0 &= \frac{\partial H}{\partial s}(q, p, r, s) \end{aligned} \quad (20)$$

with conserved quantities  $r_1, \dots, r_\ell$ , and algebraic constraints  $\frac{\partial H}{\partial s}(q, p, r, s) = 0$ . (Note that  $P_0 = \text{span}\{dr\}$  and  $G_0 = \text{span}\{\frac{\partial}{\partial s}\}$ .)

*Remark.* This form of an implicit Hamiltonian system is very close to the definition proposed by Tulczyjew [24].

Following (8) we can define the *constraint manifold*  $\mathcal{X}_c \subset \mathcal{X}$  of an implicit generalized Hamiltonian system (3) as

$$\mathcal{X}_c = \{x \in \mathcal{X} \mid dH(x) \in P_1(x)\}. \quad (21)$$

(This describes the algebraic constraints present in (3)). The implicit generalized Hamiltonian system (3) can now be reduced to an *explicit* generalized Hamiltonian system on  $\mathcal{X}_c$  provided the following Assumption is satisfied.

**ASSUMPTION 8.** *Let  $\mathcal{D}$  be a generalized Dirac structure with  $P_1$  constant-dimensional, so that  $\mathcal{D}$  can be represented as in (10). Denote  $G_0(x) = \text{Im } g(x) = \text{span}\{g_1(x), \dots, g_m(x)\}$ , with  $g_1(x), \dots, g_m(x)$  linearly independent. Assume that the  $m \times m$  matrix  $[L_{g_i}, L_{g_j}, H(x)]_{i,j=1, \dots, m}$  is invertible for all  $x \in \mathcal{X}$  satisfying  $L_{g_j} H(x) = 0$ ,  $j = 1, \dots, m$ .*

Under Assumption 8 the constraint manifold  $\mathcal{X}_c$  is given as

$$\mathcal{X}_c = \left\{ x \in \mathcal{X} \mid g^T(x) \frac{\partial H}{\partial x}(x) = 0 \right\} = \{x \in \mathcal{X} \mid L_{g_j} H(x) = 0, j = 1, \dots, m\}, \quad (22)$$

and is either empty or a submanifold of  $\mathcal{X}$  with codimension  $m$ . Consider for every  $x_c \in \mathcal{X}_c$  the canonical projection

$$P(x_c) : T_{x_c} \mathcal{X} \rightarrow T_{x_c} \mathcal{X} / G_0(x_c), \quad (23)$$

and its restriction to  $T_{x_c}\mathcal{X}_c \subset T_{x_c}\mathcal{X}$ , denoted as  $P^r(x_c)$ :

$$P^r(x_c) : T_{x_c}\mathcal{X}_c \rightarrow T_{x_c}\mathcal{X}/G_0(x_c). \quad (24)$$

We claim that  $P^r(x_c)$  is injective, and thus invertible. Indeed, let  $v \in T_{x_c}\mathcal{X}_c$  be such that  $P^r(x_c)v = 0$ , or equivalently  $v \in G_0(x_c)$ . Then  $L_v L_{g_j} H(x_c) = 0$ ,  $j = 1, \dots, m$ ,  $v \in G_0(x_c)$ , and thus by Assumption 8  $v = 0$ . Hence we may define

$$R(x_c) := [P^r(x_c)]^{-1} : T_{x_c}\mathcal{X}/G_0(x_c) \rightarrow T_{x_c}\mathcal{X}_c. \quad (25)$$

Now consider the diagram

$$T_{x_c}^*\mathcal{X}_c \xrightarrow{R^*(x_c)} \text{ann } G_0(x_c) \xrightarrow{P^*(x_c)} T_{x_c}^*\mathcal{X} \xrightarrow{J(x_c)} T_{x_c}\mathcal{X} \xrightarrow{P(x_c)} T_{x_c}\mathcal{X}/G_0(x_c) \xrightarrow{R(x_c)} T_{x_c}\mathcal{X}_c, \quad (26)$$

and define by composition the skew-symmetric mapping

$$J_c(x_c) := R(x_c)P(x_c)J(x_c)P^*(x_c)R^*(x_c). \quad (27)$$

It follows (see [21] for details) that (13) reduces to the (explicit) generalized Hamiltonian system on  $\mathcal{X}_c$  given as

$$\dot{x}_c = J_c(x_c) \frac{\partial H_c}{\partial x_c}(x_c) =: X_{H_c}(x_c) \quad (28)$$

with  $H_c : \mathcal{X}_c \rightarrow \mathbb{R}$  denoting the restriction of  $H$  to  $\mathcal{X}_c$ . Summarizing, we have obtained the following proposition.

**PROPOSITION 9.** *Let  $(\mathcal{X}, \mathcal{D}, H)$  define an implicit generalized Hamiltonian system, and let Assumption 8 be satisfied. Then  $(\mathcal{X}, \mathcal{D}, H)$  reduces to the (explicit) generalized Hamiltonian system (28) on the constraint manifold  $\mathcal{X}_c$ .*

**REMARK 10.** Proposition 9 can be also understood from the following point of view. Following the construction in Courant [5, in particular Section 1.4] we may *restrict* the generalized Dirac structure  $\mathcal{D}$  to a generalized Dirac structure  $\mathcal{D}_c$  on  $\mathcal{X}_c$  in the following manner. Let  $\mathcal{D}$  be given in representation (b). Then for  $x \in \mathcal{X}_c$  we *restrict*  $\omega(x)$  to a skew-symmetric form  $\omega_c(x)$  on the subspace  $T_x\mathcal{X}_c \cap G_1(x)$ , defining  $\mathcal{D}_c$ . Since the kernel of the skew-symmetric form  $\omega(x)$  on  $G_1(x)$  equals  $G_0(x)$ , it follows from Assumption 8 that the kernel of the form  $\omega_c(x)$  on  $T_x\mathcal{X}_c \cap G_1(x)$  is zero, and thus if we go to representation (a) of  $\mathcal{D}_c$  we obtain the dynamics (28) *without* constraints and Lagrange multipliers.

**REMARK 11.** The generalized Dirac structure as given in Proposition 7 satisfies Assumption 8 if  $H(q, p)$  is of the form  $H(q, p) = \frac{1}{2}p^T G(q)p + V(q)$  (kinetic plus potential energy), with  $G(q)$  a positive definite matrix.

The above transition from implicit to explicit generalized Hamiltonian systems becomes very transparent in case the implicit Hamiltonian system takes the form (20). Indeed, in this case Assumption 8 amounts to the symmetric matrix  $\frac{\partial^2 H}{\partial s^2}(q, p, r, s)$  being nonsingular. Hence, by the implicit function theorem applied to the last equations of



(20), one may locally express the variables  $s$  as functions of  $q, p, r$ , that is,  $s = s(q, p, r)$ . Defining the constrained Hamiltonian  $H_c(q, p, r) := H(q, p, r, s(q, p, r))$ , one then obtains the standard Hamiltonian equations of motion on the constraint manifold  $\mathcal{X}_c$  with coordinates  $q, p, r$ :

$$\begin{aligned} \dot{q} &= \frac{\partial H_c}{\partial p}(q, p, r), \\ \dot{p} &= -\frac{\partial H_c}{\partial q}(q, p, r), \\ \dot{r} &= 0. \end{aligned} \tag{29}$$

### 3. Symmetries

Following Dorfman [8], see also Courant [5], we give the following definition of an (infinitesimal) symmetry of a Dirac structure.

DEFINITION 12 [8]. Let  $\mathcal{D}$  be a generalized Dirac structure on  $\mathcal{X}$ . A vector field  $f$  on  $\mathcal{X}$  is an infinitesimal symmetry of  $\mathcal{D}$  (briefly, a *symmetry* of  $\mathcal{D}$ ) if

$$(L_f X, L_f \alpha) \in \mathcal{D}, \quad \text{for all } (X, \alpha) \in \mathcal{D}. \tag{30}$$

REMARK 13. It can be shown [8] that if  $\mathcal{D}$  is given as in Example 1 or Example 2 then  $f$  is a symmetry of  $\mathcal{D}$  iff  $L_f \{, \} = 0$ , respectively  $L_f \omega = 0$ .

REMARK 14. Analogously, we say that a *diffeomorphism*  $\varphi : \mathcal{X} \rightarrow \mathcal{X}$  is a symmetry of  $\mathcal{D}$  if

$$(\varphi_*^{-1} X, \varphi^* \alpha) \in \mathcal{D}, \quad \text{for all } (X, \alpha) \in \mathcal{D}. \tag{31}$$

Note that (31) is consistent with (30). Indeed, denote the time  $-t$  flow of the vector field  $f$  by  $\varphi_t^f : \mathcal{X} \rightarrow \mathcal{X}$ . Then  $\varphi_t^f$  is a symmetry of  $\mathcal{D}$  iff (since  $(\varphi_t^f)^{-1} = \varphi_{-t}^f$ )

$$((\varphi_{-t}^f)_* X - X, (\varphi_t^f)^* \alpha - \alpha) \in \mathcal{D}, \quad \text{for all } (X, \alpha) \in \mathcal{D}. \tag{32}$$

Thus, dividing by  $t$  and letting  $t \rightarrow 0$ ,  $(L_f X, L_f \alpha) \in \mathcal{D}$  iff  $\varphi_t^f$  is a symmetry of  $\mathcal{D}$  for all small  $t$ .

We immediately obtain the following

PROPOSITION 15. Let  $f$  be a symmetry of the generalized Dirac structure  $\mathcal{D}$ , with associated distributions  $G_0, G_1$  and co-distributions  $P_0, P_1$ . Then  $L_f G_i \subset G_i$ ,  $L_f P_i \subset P_i$ ,  $i = 0, 1$ .

*Proof:* Let  $X \in G_1$ , that is  $(X, \alpha) \in \mathcal{D}$  for some  $\alpha$ . Then by (30)  $(L_f X, L_f \alpha) \in \mathcal{D}$ , and thus  $L_f X \in G_1$ . Hence,  $L_f G_1 \subset G_1$ . Similarly,  $L_f P_1 \subset P_1$ . Since  $G_0 = \ker P_1$  and  $P_0 = \text{ann } G_1$ , it follows that  $L_f G_0 \subset G_0$ ,  $L_f P_0 \subset P_0$  (see e.g. [18, Prop. 3.46]).  $\square$

REMARK 16. If  $\mathcal{D}$  is given as in Example 3, then  $f$  is a symmetry of  $\mathcal{D}$  iff  $L_f G \subset G$ .

For implicit generalized Hamiltonian systems we obtain the following.

**PROPOSITION 17.** *Let  $(\mathcal{X}, \mathcal{D}, H)$  be an implicit generalized Hamiltonian system. Let  $f$  be a symmetry of  $\mathcal{D}$ . Moreover, let  $f$  be a symmetry of the Hamiltonian  $H : \mathcal{X} \rightarrow \mathbb{R}$ , that is,  $L_f H = 0$ . Then for all (possibly partially defined) vector fields  $X$  such that  $(X, dH) \in \mathcal{D}$  we have  $L_f X \in G_0$ . Furthermore, if Assumption 8 is satisfied, then  $f$  is tangent to the constraint manifold  $\mathcal{X}_c$ , while the restriction  $f_c$  of  $f$  to  $\mathcal{X}_c$  satisfies*

$$[f_c, X_{H_c}] = 0 \quad (33)$$

with  $X_{H_c}$  on  $\mathcal{X}_c$  defined in (28).

*Proof:* Let  $X$  be such that  $(X, dH) \in \mathcal{D}$ . Since  $f$  is a symmetry of  $\mathcal{D}$  and  $H$  we obtain  $(L_f X, L_f dH) = (L_f X, 0) \in \mathcal{D}$ , and thus  $L_f X \in G_0$ . Furthermore, since  $f$  is a symmetry of  $\mathcal{D}$  it follows from Proposition 15 that  $L_f G_0 \subset G_0$ . Hence, since  $L_f H = 0$ ,

$$L_f(L_{g_j} H) = L_{[f, g_j]} H - L_{g_j} L_f H = L_{g_j} H \quad (34)$$

for some  $g \in G_0$ , and thus  $L_f(L_{g_j} H) = 0$  on  $\mathcal{X}_c$ , implying that  $f$  is tangent to  $\mathcal{X}_c$ . By construction  $(X_{H_c}(x_c), dH(x_c)) \in \mathcal{D}(x_c)$  for all  $x_c \in \mathcal{X}_c$  (see the discussion following Assumption 8), and thus as in the first two sentences  $[f_c, X_{H_c}](x_c) \in G_0(x_c)$ . On the other hand, since  $f_c$  and  $X_{H_c}$  are vector fields on  $\mathcal{X}_c$ , their Lie bracket is also a vector field on  $\mathcal{X}_c$ . By Assumption 8 this implies that actually  $[f_c, X_{H_c}]$  is zero.  $\square$

The following subclass of symmetries of Dirac structures has been identified in [8, Theorem 7.7].

**PROPOSITION 18.** *Let  $\mathcal{D}$  be a Dirac structure on  $\mathcal{X}$  (that is, satisfying the closedness condition (2)). Let  $f$  be a vector field on  $\mathcal{X}$  for which there exists a smooth function  $F : \mathcal{X} \rightarrow \mathbb{R}$  such that  $(f, dF) \in \mathcal{D}$ . Then  $f$  is a symmetry of  $\mathcal{D}$ .*

**REMARK 19.** For a partial converse we refer to [8, Theorem 7.7].

Note, however, that the condition  $(f, dF) \in \mathcal{D}$  puts quite some restrictions on  $f$  (and  $F$ ). Indeed,  $(f, dF) \in \mathcal{D}$  implies (see (2)) that  $f \in G_1$ , and also that  $dF \in P_1$ .

The following generalization of [8, Proposition 7.3] provides a ‘‘Noether type’’ of result on the existence of conserved quantities.

**PROPOSITION 20.** *Let  $(\mathcal{X}, \mathcal{D}, H)$  be an implicit generalized Hamiltonian system with  $\mathcal{D}$  satisfying Assumption 8. Let  $f$  be a vector field on  $\mathcal{X}$  for which there exists a smooth function  $F$  such that  $(f(x), dF(x)) \in \mathcal{D}(x)$ ,  $x \in \mathcal{X}_c$ . Furthermore, let  $f$  be a symmetry for  $H$  on  $\mathcal{X}_c$ , that is  $L_f H(x) = 0$ ,  $x \in \mathcal{X}_c$ . Then  $L_{X_{H_c}}(F) = 0$  on  $\mathcal{X}_c$ , that is,  $F$  is a conserved quantity for  $X_{H_c}$  on  $\mathcal{X}_c$ .*

*Proof:* By the defining property  $\mathcal{D} = \mathcal{D}^\perp$  of a generalized Dirac structure we have

$$\langle dH(x) | f(x) \rangle + \langle dF(x) | X_{H_c}(x) \rangle = 0, \quad x \in \mathcal{X}_c,$$

since  $(f(x), dF(x)) \in \mathcal{D}(x)$ ,  $x \in \mathcal{X}_c$ , by assumption, and  $(X_{H_c}(x), dH(x)) \in \mathcal{D}(x)$ ,  $x \in \mathcal{X}_c$ , by construction.  $\square$

Now let us consider instead of a single (infinitesimal) symmetry, a symmetry Lie group  $G$  of the generalized Dirac structure  $\mathcal{D}$  on  $\mathcal{X}$ . That is to say, the Lie group  $G$  acts on  $\mathcal{X}$  by diffeomorphisms  $\Phi_g : \mathcal{X} \rightarrow \mathcal{X}, g \in G$ , see e.g. [13, 19], and  $\Phi_g$  is a symmetry of  $\mathcal{D}$  for every  $g \in G$  (see Remark 14). Equivalently, for every  $\xi \in \mathfrak{g}$  (the Lie algebra of  $G$ ) the infinitesimal generator  $X_\xi$  of the group action is an (infinitesimal) symmetry of  $\mathcal{D}$ . Throughout we assume that the quotient space  $\tilde{\mathcal{X}} := \mathcal{X}/G$  of  $G$ -orbits on  $\mathcal{X}$  is a manifold with smooth projection map  $\rho : \mathcal{X} \rightarrow \tilde{\mathcal{X}}$ . Then the generalized Dirac structure  $\mathcal{D}$  reduces to  $\tilde{\mathcal{X}}$  as follows.

**PROPOSITION 21.** *Let  $G$  be a symmetry Lie group of the generalized Dirac structure  $\mathcal{D}$  on  $\mathcal{X}$ , with quotient manifold  $\tilde{\mathcal{X}}$  and smooth projection  $\rho : \mathcal{X} \rightarrow \tilde{\mathcal{X}}$ . Then there exists a generalized Dirac structure  $\tilde{\mathcal{D}}$  on  $\tilde{\mathcal{X}}$ , called the reduced generalized Dirac structure, defined as follows*

$$(\tilde{X}, \tilde{\alpha}) \in \tilde{\mathcal{D}} \text{ if there exists } X \text{ with } \rho_* X = \tilde{X} \text{ such that } (X, \alpha) \in \mathcal{D}, \text{ where } \alpha = \rho^* \tilde{\alpha}. \quad (35)$$

Furthermore, if  $\mathcal{D}$  satisfies the closedness condition (2), then so does  $\tilde{\mathcal{D}}$ .

*Proof:* First we show that  $\tilde{\mathcal{D}}$  is a generalized Dirac structure. In order to show that  $\tilde{\mathcal{D}}^\perp \subset \tilde{\mathcal{D}}$ , let  $(\tilde{X}', \tilde{\alpha}') \in T\tilde{\mathcal{X}} \oplus T^*\tilde{\mathcal{X}}$  be such that

$$\langle \tilde{\alpha}' | \tilde{X} \rangle + \langle \tilde{\alpha} | \tilde{X}' \rangle = 0, \text{ for all } (\tilde{X}, \tilde{\alpha}) \in \tilde{\mathcal{D}}. \quad (36)$$

Now let  $X' \in T\tilde{\mathcal{X}}$  be such that  $\rho_* X' = \tilde{X}'$  and define  $\alpha' = \rho^* \tilde{\alpha}'$ . Since

$$\langle \tilde{\alpha} | \rho_* X \rangle = \langle \rho^* \tilde{\alpha} | X \rangle \quad (37)$$

for every  $\tilde{\alpha} \in T^*\tilde{\mathcal{X}}$  and every  $X \in T\tilde{\mathcal{X}}$  with  $\rho_* X$  well defined, (36) yields

$$\langle \alpha' | X \rangle + \langle \alpha | X' \rangle = 0 \quad (38)$$

for all  $(X, \alpha) \in \mathcal{D}$  such that  $\rho_* X$  is a well-defined vector field on  $\tilde{\mathcal{X}}$  and  $\alpha = \rho^* \tilde{\alpha}$  for some  $\tilde{\alpha} \in T^*\tilde{\mathcal{X}}$ . Since  $G$  is a symmetry group of  $\mathcal{D}$ , it follows from Remark 14 that

$$(X, \alpha) \in \mathcal{D} \Rightarrow (\Phi_{g^{-1}} X, (\Phi_g^*)^{-1} \alpha) \in \mathcal{D}, \quad \forall g \in G. \quad (39)$$

Thus, (38) also holds for all  $(X, \alpha) \in \mathcal{D}$  such that  $\alpha = \rho^* \tilde{\alpha}$  for some  $\tilde{\alpha} \in T^*\tilde{\mathcal{X}}$ . Hence,

$$(X', \alpha') \in (\mathcal{D} \cap C)^\perp = \mathcal{D} + C^\perp, \quad (40)$$

with  $C$  denoting the vector subbundle of  $T\mathcal{X} \oplus T^*\mathcal{X}$  spanned by all  $(X, \alpha)$  such that  $\alpha = \rho^* \tilde{\alpha}$  for some  $\tilde{\alpha} \in T^*\tilde{\mathcal{X}}$ , and where we have used  $\mathcal{D}^\perp = \mathcal{D}$  ( $\mathcal{D}$  is a generalized Dirac structure). We claim that

$$C^\perp = \{(\tilde{X}, 0) | \rho_* \tilde{X} = 0\}. \quad (41)$$

Indeed, the inclusion  $\supset$  is obvious, while for the reverse inclusion we note that if  $(\tilde{X}, \tilde{\alpha})$  is such that  $\langle \tilde{\alpha} | X \rangle + \langle \alpha | \tilde{X} \rangle = 0$  for all  $(X, \alpha) \in C$ , then (taking  $X = 0$ )

$\langle \alpha | \tilde{X} \rangle = 0$  for all  $\alpha = \rho^* \tilde{\alpha}$ ,  $\tilde{\alpha} \in T^* \tilde{\mathcal{X}}$ , and thus  $\rho_* \tilde{X} = 0$ . Hence,  $0 = \langle \tilde{\alpha} | X \rangle + \langle \alpha | \tilde{X} \rangle = \langle \tilde{\alpha} | X \rangle$  for all  $X$ , implying  $\tilde{\alpha} = 0$ . Therefore, by (40) and (41) there exists a vector field  $\tilde{X}$  with  $\rho_* \tilde{X} = 0$  such that  $(X' + \tilde{X}, \alpha') \in \mathcal{D}$ . Since  $\rho_*(X' + \tilde{X}) = \rho_* X' = \tilde{X}'$ , this implies  $(\tilde{X}', \alpha') \in \tilde{\mathcal{D}}$ , showing that  $\tilde{\mathcal{D}}^\perp \subset \tilde{\mathcal{D}}$ . The reverse inclusion  $\tilde{\mathcal{D}} \subset \tilde{\mathcal{D}}^\perp$  follows easily.

Finally, let  $\mathcal{D}$  be closed. Take  $(\tilde{X}_i, \tilde{\alpha}_i) \in \tilde{\mathcal{D}}$ ,  $i = 1, 2, 3$ , that is,  $(X_i, \rho^* \tilde{\alpha}_i) \in \mathcal{D}$ ,  $\rho_* X_i = \tilde{X}_i$ ,  $i = 1, 2, 3$ . Using the general equality (37) and  $\rho^*(L_{\rho_* X} \tilde{\alpha}) = L_X \rho^* \tilde{\alpha}$  we obtain

$$\begin{aligned} \langle L_{\tilde{X}_1} \tilde{\alpha}_2 | \tilde{X}_3 \rangle + \langle L_{\tilde{X}_2} \tilde{\alpha}_3 | \tilde{X}_1 \rangle + \langle L_{\tilde{X}_3} \tilde{\alpha}_1 | \tilde{X}_2 \rangle \\ = \langle L_{\tilde{X}_1} \tilde{\alpha}_2 | \rho_* X_3 \rangle + \langle L_{\tilde{X}_2} \tilde{\alpha}_3 | \rho_* X_1 \rangle + \langle L_{\tilde{X}_3} \tilde{\alpha}_1 | \rho_* X_2 \rangle \\ = \langle L_{X_1} \rho^* \tilde{\alpha}_2 | X_3 \rangle + \langle L_{X_2} \rho^* \tilde{\alpha}_3 | X_1 \rangle + \langle L_{X_3} \rho^* \tilde{\alpha}_1 | X_2 \rangle = 0, \end{aligned}$$

since  $\mathcal{D}$  satisfies (2). Hence, also  $\tilde{\mathcal{D}}$  is closed.  $\square$

Next question is how we can effectively compute the reduced (generalized) Dirac structure  $\tilde{\mathcal{D}}$  from  $\mathcal{D}$ . We will only do this under the following

**ASSUMPTION 22.** *The co-distribution  $P_1$  of the generalized Dirac structure  $\mathcal{D}$  on  $\mathcal{X}$  is constant-dimensional. Denote by  $V$  the distribution on  $\mathcal{X}$  tangent to the orbits of  $G$  (that is, spanned by the infinitesimal symmetries). The co-distribution  $P_1 \cap \text{ann } V$  is also constant-dimensional.*

By Theorem 6 the generalized Dirac structure  $\mathcal{D}$  on  $\mathcal{X}$  can now be represented as in (10). Then define the reduced skew-symmetric linear map

$$J(x) : P_1(x) \cap \text{ann } V(x) \longrightarrow (P_1(x) \cap \text{ann } V(x))^* \simeq T_x \mathcal{X} / (G_0(x) + V(x)) \quad (42)$$

by simple restriction of  $J(x)$  to  $P_1(x) \cap \text{ann } V(x)$ . Since  $J(x_1) = J(x_2)$  for all  $x_1, x_2$  with  $\rho(x_1) = \rho(x_2)$ ,  $\tilde{\mathcal{D}}$  can be seen to be given as in (10), that is

$$\tilde{\mathcal{D}}(\tilde{x}) = \{(\tilde{X}, \tilde{\alpha}) \mid \tilde{X}(\tilde{x}) - \tilde{J}(\tilde{x})\tilde{\alpha}(\tilde{x}) \in \ker \tilde{P}_1(\tilde{x}), \tilde{x} \in \tilde{\mathcal{X}}, \tilde{\alpha} \in \tilde{P}_1\}. \quad (43)$$

with  $\tilde{J} \circ \rho = J$ , and  $\tilde{P}_1$  the reduced constant-dimensional co-distribution on  $\mathcal{X}$  defined as

$$\tilde{P}_1 = \text{span}\{\tilde{\alpha} \mid \rho^* \tilde{\alpha} \in P_1\}. \quad (44)$$

(Note that  $\rho^* \tilde{\alpha}$  is zero on  $V$ .)

Based on Proposition 21, we immediately obtain the following result on reduction of implicit generalized Hamiltonian systems.

**PROPOSITION 23.** *Let  $(\mathcal{X}, \mathcal{D}, H)$  be an implicit generalized Hamiltonian system. Let  $G$  be a symmetry Lie group of the generalized Dirac structure  $\mathcal{D}$  on  $\mathcal{X}$ , with quotient manifold  $\tilde{\mathcal{X}}$ , smooth projection  $\rho : \mathcal{X} \rightarrow \tilde{\mathcal{X}}$ , and reduced generalized Dirac structure  $\tilde{\mathcal{D}}$  on  $\tilde{\mathcal{X}}$  as in Proposition 21. Furthermore, suppose the action of  $G$  on  $\mathcal{X}$  leaves  $H$  invariant, leading to a reduced Hamiltonian  $\tilde{H} : \tilde{\mathcal{X}} \rightarrow \mathbb{R}$  such that  $H = \tilde{H} \circ \rho$ . Then the implicit generalized Hamiltonian system  $(\mathcal{X}, \mathcal{D}, H)$  projects to the implicit generalized Hamiltonian system  $(\tilde{\mathcal{X}}, \tilde{\mathcal{D}}, \tilde{H})$ .*

*Proof:* By definition of  $\tilde{D}$  we have (note that  $\rho^*d\tilde{H} = dH$ )

$$(\tilde{X}(\tilde{x}), d\tilde{H}(\tilde{x})) \in \tilde{D}(\tilde{x}) \iff (X(x), dH(x)) \in \mathcal{D}(x)$$

for some  $X$  with  $\rho_*X = \tilde{X}$  and all  $x \in \mathcal{X}$  such that  $\rho(x) = \tilde{x}$ . Substituting  $\tilde{x}$  for  $\tilde{X}(\tilde{x})$ , and  $\dot{x}$  for  $X(x)$  we obtain the result.  $\square$

Finally, let us now specialize the theory of symmetries of implicit generalized Hamiltonian systems to the systems arising from mechanical systems subject to kinematic constraints, as described in Example 4 and formalized in Proposition 7. First, we may identify the following important class of symmetries of the underlying generalized Dirac structure.

**PROPOSITION 24.** *Consider the generalized Dirac structure  $\mathcal{D}$  on  $T^*Q$  given in Proposition 7. Let  $f$  be a vector field on  $T^*Q$  satisfying  $L_f\omega = 0$  and  $L_fP_0 \subset P_0$ . Then  $f$  is a symmetry of  $\mathcal{D}$ .*

*Proof:* Let  $(X, \alpha) \in \mathcal{D}$ . Then  $\alpha = i_X\omega + \beta$ , with  $\beta \in P_0$ . Thus, since  $L_f\omega = 0$ ,

$$(L_fX, L_f\alpha) = (L_fX, i_{L_fX}\omega + L_f\beta)$$

which is again in  $\mathcal{D}$ , since  $L_f\beta \in L_fP_0 \subset P_0$ .  $\square$

From the property  $L_f\omega = 0$  it follows, see e.g. [11, 13], that at least locally there exists a function  $F$  such that  $i_f\omega = dF$ . (Thus  $f$  is a (locally) Hamiltonian vector field on  $T^*Q$  with respect to the natural symplectic form  $\omega$  on  $T^*Q$ , and Hamiltonian  $F$ ). In view of Proposition 20 one may thus wonder when the additional condition  $L_fH = 0$  on  $\mathcal{X}_c$  implies that  $F$  is a conserved quantity for the constrained Hamiltonian system corresponding to  $\mathcal{D}$  and  $H$ . This is answered in the following proposition.

**PROPOSITION 25.** *Consider the generalized Dirac structure  $\mathcal{D}$  on  $T^*Q$  given in Proposition 7, satisfying Assumption 8. Let  $f$  be a Hamiltonian vector field on  $T^*Q$ , that is,  $i_f\omega = dF$  for some  $F : T^*Q \rightarrow \mathbb{R}$ . Additionally, let  $f$  satisfy  $L_fH(x) = 0$ ,  $x \in \mathcal{X}_c$ . Then  $L_{X_{H_c}}(F) = 0$  on  $\mathcal{X}_c$  if  $f(x) \in G_1(x)$ ,  $x \in \mathcal{X}_c$ .*

*Proof:* Following Proposition 20 we only have to show that  $(f(x), dF(x)) \in \mathcal{D}(x)$ ,  $x \in \mathcal{X}_c$ . However, this is obvious from the assumption  $f(x) \in G_1(x)$ ,  $x \in \mathcal{X}_c$ , since  $i_f\omega = dF$  and  $\mathcal{D}$  is given as in representation (b) of Theorem 6.  $\square$

Usually, a symmetry  $f$  as in Propositions 24 and 25 occurs by first considering a vector field  $f_Q$  on the configuration manifold  $Q$  which leaves the constraint codistribution  $P_Q := \text{span}\{\alpha_1, \dots, \alpha_k\}$  on  $Q$  invariant, that is  $L_{f_Q}P_Q \subset P_Q$ . Then the vector field  $f_Q$  naturally lifts to a vector field  $f$  on  $T^*Q$  which satisfies  $L_f\omega = 0$  and  $L_fP_0 \subset P_0$ . (In fact,  $f$  is defined as the Hamiltonian vector field on  $T^*Q$  with respect to  $\omega$  and the Hamiltonian  $F(q, p) := p^T f_Q(q)$ .) This is precisely the class of “symmetries of nonholonomic mechanical systems” as treated in [2] and, within

a Lagrangian framework, in [4, 9]. (Note that the class considered in [1] is more restrictive.) As in these references and as above, we then look at symmetry groups that are defined by a Lie group  $G$  acting on  $T^*Q$  by canonical transformations and leaving the co-distribution  $P_0$  invariant (or acting on  $Q$  and leaving  $P_Q$  invariant).

#### 4. Examples

The theory of Section 3 will be illustrated on three simple examples. The first two examples have been treated before in [2, 4] and concern mechanical systems with nonholonomic constraints as formalized in Proposition 7, while the last example is concerned with a simple  $LC$  electrical circuit.

##### Example 4.1. Motion of a particle subject to a nonholonomic constraint ([2])

Consider a particle in  $\mathbb{R}^3$  with kinetic energy  $\frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$  subject to the nonholonomic constraint  $\dot{z} = y\dot{x}$ . In the formulation of Proposition 7 this means that  $Q = \mathbb{R}^3$ , with coordinates  $(x, y, z)$ ,  $T^*Q = \mathbb{R}^3 \times \mathbb{R}^3$  with canonical coordinates  $(x, y, z, p_x, p_y, p_z)$ , and the generalized Dirac structure  $\mathcal{D}$  on  $T^*Q$  is defined as in (11) for

$$P_0 = \text{span}\{dz - ydx\}. \quad (45)$$

After Legendre transformation the Hamiltonian (total energy) is given by

$$H(x, y, z, p_x, p_y, p_z) = \frac{1}{2}(p_x^2 + p_y^2 + p_z^2). \quad (46)$$

Clearly, the generalized Dirac structure  $\mathcal{D}$  as well as the Hamiltonian  $H$  are invariant under translations of the  $x$ - and  $z$ -coordinates, so that (cf. Assumption 22)

$$V = \text{span}\left\{\frac{\partial}{\partial x}, \frac{\partial}{\partial z}\right\}. \quad (47)$$

(Note that  $\frac{\partial}{\partial x}$  and  $\frac{\partial}{\partial z}$  are vector fields as in Proposition 24.) We compute

$$\begin{aligned} G_0 &= \text{span}\left\{\frac{\partial}{\partial p_z} - y\frac{\partial}{\partial p_x}\right\}, \\ P_1 &= \text{span}\{dx, dy, dz, ydp_z + dp_x, dp_y\}, \\ P_1 \cap \text{ann } V &= \text{span}\{dy, ydp_z + dp_x, dp_y\}. \end{aligned} \quad (48)$$

The reduced space  $\tilde{\mathcal{X}}$  is given by  $\mathbb{R}^4$  with coordinates  $(y, p_x, p_y, p_z)$ , whereas

$$\tilde{P}_1 = \text{span}\{dy, ydp_z + dp_x, dp_y\}, \quad (49)$$

and thus

$$\ker \bar{P}_1 = \bar{G}_0 = \text{span} \left\{ y \frac{\partial}{\partial p_x} - \frac{\partial}{\partial p_z} \right\}. \quad (50)$$

It follows that the *reduced* implicit generalized Hamiltonian system on  $\bar{\mathcal{X}}$  (see Proposition 23) is given in representation (a) as

$$\begin{bmatrix} \dot{y} \\ \dot{p}_x \\ \dot{p}_y \\ \dot{p}_z \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ p_x \\ p_y \\ p_z \end{bmatrix} + \begin{bmatrix} 0 \\ y \\ 0 \\ -1 \end{bmatrix} \lambda, \quad (51)$$

$$0 = yp_x - p_z.$$

Note furthermore that  $\bar{P}_0 = \text{span}\{ydp_z + dp_x\}$ , representing the (again nonintegrable) constraint  $yp_z + p_x = 0$ . (This is, however, not anymore a *kinematic* constraint!) Since neither  $\frac{\partial}{\partial x}$  nor  $\frac{\partial}{\partial z}$  are contained in  $G_1 = \ker P_0$ , Proposition 25 does not yield first integrals for the constrained system on  $\mathcal{X}_c$ .

On the other hand, we can easily eliminate the constraint  $0 = yp_x - p_z$  and the multiplier  $\lambda$  from (51) leading to

$$\begin{cases} \dot{y} = p_y, \\ \dot{p}_y = 0, \\ \dot{p}_x = -\frac{y}{1+y^2} p_x p_y. \end{cases} \quad (52)$$

The last differential equation can be solved as  $p_x = \frac{c}{\sqrt{1+y^2}}$ ,  $c \in \mathbb{R}$ , leading to the same solutions as obtained in [2].

**Example 4.2. The rolling penny** (see e.g. [4, 22]).

Consider a vertical wheel rolling without slipping on a horizontal plane. Let  $x, y$  be the Cartesian coordinates of the point of contact of the wheel with the plane. Furthermore,  $\theta$  denotes the rotation angle of the wheel, and  $\varphi$  the heading angle on the plane. The rolling constraints  $\dot{x} - \dot{\theta} \cos \varphi = 0$ ,  $\dot{y} - \dot{\theta} \sin \varphi = 0$  are nonholonomic. In the formulation of Proposition 7 we have  $Q = \mathbb{R}^2 \times S^1 \times S^1$  with coordinates  $(x, y, \theta, \varphi)$ ,  $T^*Q$  with canonical coordinates  $(x, y, \theta, \varphi, p_x, p_y, p_\theta, p_\varphi)$ , and the generalized Dirac structure  $\mathcal{D}$  on  $T^*Q$  is defined as in (11) for

$$P_0 = \text{span}\{dx - \cos \varphi d\theta, dy - \sin \varphi d\theta\}. \quad (53)$$

The Hamiltonian (setting all parameters equal to 1) is given as  $H(x, y, \theta, \varphi, p_x, p_y, p_\theta, p_\varphi) = \frac{1}{2}(p_x^2 + p_y^2 + p_\theta^2 + p_\varphi^2)$ . The generalized Dirac structure  $\mathcal{D}$  as well as the Hamiltonian  $H$  are invariant under translation of the  $x$ - and  $y$ -coordinates, and rotation of the  $\theta$ -coordinate, so that

$$V = \text{span} \left\{ \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial \theta} \right\}. \quad (54)$$

We compute

$$\begin{aligned} G_0 &= \text{span} \left\{ \frac{\partial}{\partial p_x} - \cos \varphi \frac{\partial}{\partial p_\theta}, \frac{\partial}{\partial p_y} - \sin \varphi \frac{\partial}{\partial p_\theta} \right\}, \\ P_1 &= \text{span}\{dx, dy, d\theta, d\varphi, \cos \varphi dp_x + \sin \varphi dp_y + dp_\theta, dp_\varphi\}. \end{aligned} \quad (55)$$

The reduced space  $\tilde{\mathcal{X}}$  has coordinates  $(\varphi, p_x, p_y, p_\theta, p_\varphi)$  with

$$\tilde{P}_1 = \text{span}\{d\varphi, \cos \varphi dp_x + \sin \varphi dp_y + dp_\theta, dp_\varphi\}, \quad (56)$$

and thus

$$\tilde{G}_0 = \ker \tilde{P}_1 = \text{span} \left\{ \frac{\partial}{\partial p_x} - \cos \varphi \frac{\partial}{\partial p_\theta}, \frac{\partial}{\partial p_y} - \sin \varphi \frac{\partial}{\partial p_\theta} \right\}, \quad (57)$$

leading to the reduced implicit generalized Hamiltonian system

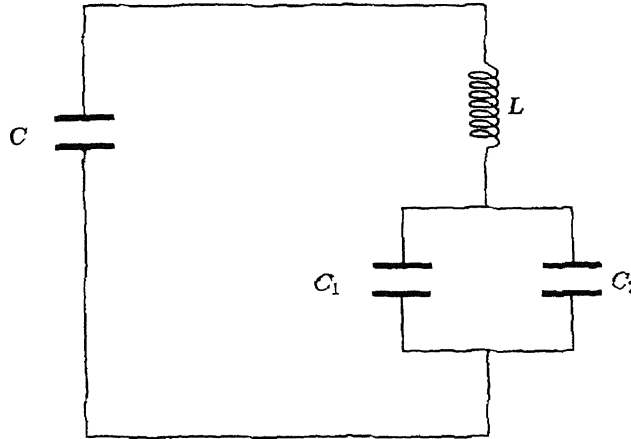
$$\begin{aligned} \begin{bmatrix} \dot{\varphi} \\ \dot{p}_x \\ \dot{p}_y \\ \dot{p}_\theta \\ \dot{p}_\varphi \end{bmatrix} &= \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ p_x \\ p_y \\ p_\theta \\ p_\varphi \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ -\cos \varphi & -\sin \varphi \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}, \\ 0 &= p_x - \cos \varphi \cdot p_\theta, \\ 0 &= p_y - \sin \varphi \cdot p_\theta. \end{aligned} \quad (58)$$

Furthermore, one computes  $\tilde{P}_0 = \text{span}\{\cos \varphi dp_x + \sin \varphi dp_y + dp_\theta\}$  representing the (nonintegrable) constraint  $\cos \varphi \cdot \dot{p}_x + \sin \varphi \cdot \dot{p}_y + \dot{p}_\theta = 0$ .

This example can be modified in a number of directions by adding to the Hamiltonian  $H$  potential energy terms depending on  $x$  and/or  $y$  (inclined versus horizontal plane), or depending on  $\theta$  (a torsional spring attached to the wheel). For instance, by adding a potential energy  $H_{\text{pot}}(\theta)$ , the symmetry distribution becomes  $V = \text{span}\{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\}$ , and  $\tilde{P}_0 = 0$  on the reduced space with coordinates  $\{\theta, \varphi, p_x, p_y, p_\theta, p_\varphi\}$ .



**Example 4.3.** Consider the following  $LC$  circuit



where the most left capacitor  $C$  represents a large (parasitic) capacitance. Using Kirchhoff's laws, the dynamics is described by the differential-algebraic equations

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\varphi} \\ \dot{q}_1 \\ \dot{q}_2 \\ \dot{q} \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & -1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial H}{\partial \varphi} \\ \frac{\partial H}{\partial q_1} \\ \frac{\partial H}{\partial q_2} \\ \frac{\partial H}{\partial q} \end{bmatrix} \quad (59)$$

with  $\varphi$  the magnetic flux of the inductor,  $q_1, q_2$  and  $q$  the electric charges of the capacitors  $C_1, C_2$  and  $C$ , and  $H(\varphi, q_1, q_2, q) = \frac{1}{2L}\varphi^2 + \frac{1}{2C_1}q_1^2 + \frac{1}{2C_2}q_2^2 + \frac{1}{2C}q^2$  the total (magnetic and electric) energy (for simplicity assumed to be quadratic). This describes an implicit Hamiltonian system on  $\mathbb{R}^4$ , with the Dirac structure solely determined by Kirchhoff's laws, that is, by the two square matrices in (59) (see [20, 7] for further details). In the limit  $C \rightarrow \infty$  (corresponding to short-circuiting the most left branch of the circuit), the system admits the infinitesimal symmetry  $\frac{\partial}{\partial q}$ , and the system reduces to the following implicit Hamiltonian system on the reduced space  $\mathbb{R}^3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} \dot{\varphi} \\ \dot{q}_1 \\ \dot{q}_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & -1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial \hat{H}}{\partial \varphi} \\ \frac{\partial \hat{H}}{\partial q_1} \\ \frac{\partial \hat{H}}{\partial q_2} \end{bmatrix} \quad (60)$$

with  $\hat{H}(\varphi, q_1, q_2) = \frac{1}{2L}\varphi^2 + \frac{1}{2C_1}q_1^2 + \frac{1}{2C_2}q_2^2$ .

## 5. Conclusions

After a brief exposé of generalized Dirac structures and implicit generalized Hamiltonian systems, including the special case of mechanical systems subject to kinematic constraints, we have shown how the notion of symmetry of Dirac structures as proposed in [8] can be naturally used for the study of implicit (generalized) Hamiltonian systems with symmetry. The main results concern the *reduction* of the (generalized) Dirac structure and the implicit Hamiltonian system to the quotient manifold of the orbits of the symmetry group. Some results concerning the existence of conserved quantities (first integrals) have been also derived.

We hope to have demonstrated that the use of Dirac structures offers a conceptually clear approach to handle implicit Hamiltonian systems with symmetry, even for the special case of mechanical systems with nonholonomic constraints as already treated in [2, 6, 10]; see [9, 4] for the Lagrangian picture.

Clearly, many aspects of implicit Hamiltonian systems with symmetry have not been covered in this brief paper. Especially the further reduction using first integrals and its relation with the structure of the group action (see e.g. [14, 13, 19, 1, 11] for the “standard” Hamiltonian case) should be a topic for further research.

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